

Asymptotics of the counting function of k -th power-free elements in an arithmetic semigroup

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ABSTRACT. For any $k \geq 2$, we find the asymptotics of the counting function of k -th power-free elements in an additive arithmetic semigroup with exponential growth of the abstract prime counting function. This paper continues the authors' earlier research dealing with the case of $k = \infty$.

1. Introduction and statement of the problem

Let G be an additive arithmetical semigroup [1, pp. 11, 56]; i.e.,

(i) G is a commutative semigroup with identity element 1.

(ii) There exists a (uniquely determined) countable subset $P \subset G$ (whose elements are called the *primes* of G) such that every element $a \in G$, $a \neq 1$, has a factorization of the form

$$(1) \quad a = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$$

with some positive integers s, n_1, \dots, n_s and elements $p_1, \dots, p_s \in P$, and this factorization is unique up to the order of factors.

(iii) A mapping $\partial: G \rightarrow \mathbf{R}$ (called the *degree mapping*) is given such that $\partial(1) = 0$, $\partial(p) > 0$ for all $p \in P$, $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$, and the number $\mathcal{N}_G^\#(x)$ of elements $a \in G$ such that $\partial(a) \leq x$ is finite for every $x > 0$.

For an integer $k \geq 2$, an element $a \in G$ is said to be *k -th power-free* if it has no divisors of the form b^k , where $1 \neq b \in G$. We denote the set

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of k -th power-free elements in G by $G_k \subset G$ and the number of k -th power-free elements of degree $\leq x$ by $\mathcal{N}_{G,k}^\#(x)$. It is natural to extend the definition to $k = \infty$ by setting $G_\infty = G$; then $\mathcal{N}_{G,\infty}^\#(x) = \mathcal{N}_G^\#(x)$.

Our aim is to find the asymptotics of $\mathcal{N}_{G,k}^\#(x)$ as $x \rightarrow \infty$ under the assumption that the number $\pi_G^\#(x)$ of primes $p \in P$ such that $\partial(p) \leq x$ has the asymptotics

$$(2) \quad \pi_G^\#(x) = \rho x^\gamma e^x (1 + O(x^{-\delta})), \quad x \rightarrow \infty,$$

for some $\rho > 0$, $\gamma > -1$, and $\delta \in (0, 1]$. The limit case of $k = \infty$ was considered in [2–4] (where one can also find more detailed bibliographical remarks). Here we essentially show that the results obtained there remain valid, *mutatis mutandis*, for the case of finite k .

Theorems deriving the asymptotic behavior of $\mathcal{N}_G^\#(x)$ as $x \rightarrow \infty$ from that of $\pi_G^\#(x)$ are known as (*inverse*) *abstract prime number theorems*, and the corresponding theorems for $\mathcal{N}_{G,k}^\#(x)$ are a generalization of these. Apart from the purely number-theoretic meaning, the function $\mathcal{N}_{G,k}^\#(x)$ has a natural interpretation in statistical mechanics. Let us enumerate the elements of P in some way, $P = \{p_1, p_2, \dots\}$, and set $\lambda_j = \partial(p_j)$. Then $\mathcal{N}_{G,k}^\#(x)$ is the number of solutions of the inequality

$$(3) \quad \sum_{j=1}^{\infty} \lambda_j n_j \leq x$$

in integers n_j such that

$$(4) \quad 0 \leq n_j < k.$$

Inequality (3) describes the states with total energy $\leq x$ of a system of noninteracting indistinguishable particles, n_j being the number of particles at the energy level j with energy λ_j . Inequalities (4) imply that there are at most $k - 1$ particles at each energy level. In other words, the particles obey the Gentile statistics (see [5–7], [8, p. 258]), which becomes the well-known Bose–Einstein statistics (any number of particles at any level) and Fermi–Dirac statistics (at most one particle at each level) in the limit cases of $k = \infty$ and $k = 2$, respectively. The logarithm $\ln \mathcal{N}_{G,k}^\#(x)$ is the entropy of the system. Note, however, that the counting function $\pi_G^\#(x)$ usually has a power-law asymptotics rather than the exponential asymptotics (2) in statistical mechanics, at least if the individual particles have finitely many degrees of freedom (e.g., see [9] and the survey [10], where further references can be found). Our interest in the asymptotics (2) is partly motivated by problems arising when calculating the number of localized Gaussian packets in the theory of dynamical systems on metric and decorated graphs [11–13], where exponential growth is associated with positivity of topological entropy of the manifolds in question (see [14–16]).

2. Main results

Assume that condition (2) is satisfied. The Dirichlet series

$$(5) \quad \zeta_{G,k}(s) = \sum_{a \in G_k} e^{-\partial(a)s}, \quad s = \sigma + it,$$

converges absolutely in the half-plane $\sigma > 1$, and one has the Euler identity

$$(6) \quad \zeta_{G,k}(s) = \prod_{p \in P} \frac{1 - e^{-\partial(p)sk}}{1 - e^{-\partial(p)s}}.$$

The proof is the same as for the zeta function $\zeta_G(s)$ of G (e.g., see [1, p. 36]), which is the special case of (5) for $k = \infty$.

Now we are in a position to state the main results of the paper.

THEOREM 1. *Under condition (2), the function $\mathcal{N}_{G,k}^\#(x)$ has the following asymptotics as $x \rightarrow \infty$:*

$$(7) \quad \mathcal{N}_{G,k}^\#(x) = \frac{e^{xs} \zeta_{G,k}(s)}{\sqrt{2\pi(\ln \zeta_{G,k}(s))''}} \Big|_{s=\beta(x)} (1 + O(x^{-\kappa})),$$

where $s = \beta(x) > 1$ is the unique real solution of the equation

$$(8) \quad x + (\ln \zeta_{G,k}(s))' = 0$$

and $\kappa > 0$ is an arbitrary number such that

$$(9) \quad \kappa < \frac{\delta}{2 + \gamma}, \quad \kappa \leq \frac{1 + \gamma}{2 + \gamma}.$$

Theorem 1 gives the asymptotics of $\mathcal{N}_{G,k}^\#(x)$ in terms of the function $\zeta_{G,k}(s)$, which is itself given by the infinite product (6). The formulas for the logarithmic asymptotics are much simpler and depend only on the constants ρ , γ , and δ occurring in the asymptotics (2) of the prime counting function. Namely, the following theorem holds.

THEOREM 2. *Under condition (2), the function $\ln \mathcal{N}_{G,k}^\#(x)$ has the following asymptotics as $x \rightarrow \infty$:*

$$(10) \quad \ln \mathcal{N}_{G,k}^\#(x) = x + 2(\rho\Gamma(\gamma + 2))^{\frac{1}{\gamma+2}} x^{\frac{\gamma+1}{\gamma+2}} + R(x)$$

if $\delta \leq \min\{1, 1 + \gamma\}$, where

$$R(x) = O(x^{\frac{\gamma+1-\delta}{\gamma+2}}) \quad \text{if } \delta < 1 + \gamma \quad \text{and} \quad R(x) = O(\ln x) \quad \text{if } \delta = 1 + \gamma;$$

$$(11) \quad \ln \mathcal{N}_{G,k}^\#(x) = x + 2(\rho\Gamma(\gamma + 2))^{\frac{1}{\gamma+2}} x^{\frac{\gamma+1}{\gamma+2}} - \frac{1}{2} \frac{\gamma + 3}{\gamma + 2} \ln x + O(1)$$

if $1 \geq \delta > 1 + \gamma$.

REMARK 1. In contrast to the asymptotics obtained in Theorem 1, the logarithmic asymptotics provided by Theorem 2 does not feel the difference between the cases of $k = \infty$ and finite k .

REMARK 2. Similar results were obtained in [17] in a different setting. (The analysis in that paper only applies to the case in which the mapping ∂ is integer-valued.)

3. Proof of the theorems

The proof of both theorems is completely similar to that given in [4] for the case of $k = \infty$, and here we only give a brief outline of the reasoning. The argument relies on asymptotic formulas for the function $\ln \zeta_{G,k}(\sigma)$ and its derivatives as $\sigma \downarrow 1$. These formulas have the form

$$(12) \quad \ln \zeta_{G,k}(\sigma) = \rho \Gamma(1 + \gamma)(\sigma - 1)^{-1-\gamma}(1 + o(1)),$$

$$(13) \quad (\ln \zeta_{G,k}(\sigma))' = -\rho \Gamma(\gamma + 2)(\sigma - 1)^{-\gamma-2}(1 + o(1)),$$

$$(14) \quad (\ln \zeta_{G,k}(\sigma))'' = \rho \Gamma(\gamma + 3)(\sigma - 1)^{-\gamma-3}(1 + o(1))$$

(we only write out the leading terms of the asymptotics) and coincide modulo $O(1)$ with those obtained in [4] for $\ln \zeta_G(\sigma)$, because

$$\ln \zeta_{G,k}(\sigma) - \ln \zeta_G(\sigma) = \sum_{p \in P} \ln(1 - e^{-\partial(p)\sigma k}) = O(1) \quad \text{as } \sigma \downarrow 1.$$

(Recall that $k \geq 2$.)

OUTLINE OF PROOF OF THEOREM 1. We have

$$(15) \quad \mathcal{N}_{G,k}^\#(x) = \sum_{a \in G_k} H\left(\frac{x - \partial(a)}{\varepsilon}\right),$$

where $H(x)$ is the Heaviside step function and $\varepsilon > 0$ is arbitrary. Take smooth functions $\chi_\pm(x)$ such that

$$(16) \quad \chi_-(x) \leq H(x) \leq \chi_+(x) \quad \text{for all } x, \quad \chi_\pm(x) = H(x), \quad |x| \geq 1;$$

then

$$\sum_{a \in G_k} \chi_-\left(\frac{x - \partial(a)}{\varepsilon}\right) \leq \mathcal{N}_{G,k}^\#(x) \leq \sum_{a \in G_k} \chi_+\left(\frac{x - \partial(a)}{\varepsilon}\right).$$

Using the generalization in [4, Proposition 3] of the well-known Perron formula [18, p. 12, Theorem 13], we obtain

$$(17) \quad I_-(x, \varepsilon) \leq \mathcal{N}_{G,k}^\#(x) \leq I_+(x, \varepsilon),$$

where

$$(18) \quad I_\pm(x, \varepsilon) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} \zeta_{G,k}(s) \varepsilon \tilde{\chi}_\pm(\varepsilon s) ds,$$

$\tilde{\chi}_\pm(s)$ is the two-sided Laplace transform of $\chi_\pm(x)$, and the integrals are independent of the choice of $\sigma > 1$.

Now we compute these integrals by the mountain pass method [19, Ch. 4, p. 170]. The phase function is

$$(19) \quad S(x, s) = xs + \ln \zeta_{G,k}(s),$$

and the amplitude is $\varepsilon \tilde{\chi}_{\pm}(\varepsilon s)$. The equation $\partial S(x, s)/\partial s = 0$ for the stationary points of the phase function (19) coincides with (8) and has a unique real solution $s = \beta(x) > 1$ for each $x > 0$. Further, $\beta(x) \rightarrow 1$ as $x \rightarrow \infty$, and it follows from (13) that

$$(20) \quad \beta(x) - 1 \sim Cx^{-1/(\gamma+2)}, \quad C = (\rho\Gamma(\gamma+2))^{\frac{1}{\gamma+2}}.$$

In (18), we take the integration contour to be given by $\sigma = \beta(x)$; this is a mountain pass contour for this integral. We make a change of the integration variable s by the formula $s = \beta(x) + i\xi$, so that the contour of integration with respect to the new variable ξ coincides with the real line. Further, set

$$(21) \quad x = x(\beta) \equiv -(\ln \zeta_{G,k}(\beta))';$$

this is the inverse function of $\beta(x)$. These transforms give the integrals

$$(22) \quad I_{\pm}(x(\beta), \varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{S(x(\beta), \beta+i\xi)} \varepsilon \tilde{\chi}_{\pm}(\varepsilon(\beta+i\xi)) d\xi,$$

with the saddle point $\xi = 0$. By using the Mountain Pass Theorem [19, Ch. 4, Theorem 1.3, p. 170] and the estimates in [4] for the phase function $\Phi(\beta, \xi) = S(x(\beta), \beta+i\xi)$ and the amplitude $\varphi_{\pm}(\beta, \xi) = \varepsilon(\beta) \tilde{\chi}_{\pm}(\varepsilon(\beta)(\beta+i\xi))$ with $\varepsilon(\beta) = (\beta-1)^{\kappa(2+\gamma)}$, $\kappa > 0$, we finally obtain

$$(23) \quad \begin{aligned} I_{\pm}(x(\beta), \varepsilon(\beta)) &= \frac{e^{x(\beta)\beta} \zeta_{G,k}(\beta)}{\sqrt{2\pi(\ln \zeta_{G,k}(\beta))''}} \\ &\quad \times (1 + O(\varepsilon(\beta)) + O(\beta-1) + O((\beta-1)^{1+\gamma})), \end{aligned}$$

which, together with (17), gives (7). \square

OUTLINE OF PROOF OF THEOREM 2. This theorem follows if one takes the logarithm of both sides of formula (7) in Theorem 1 and then uses the asymptotics (20) of $\beta(x)$ together with the asymptotics of $\ln \zeta_{G,k}(\sigma)$ and $(\ln \zeta_{G,k}(\sigma))''$ whose leading terms are given by (12) and (14), respectively. \square

4. Simulation results

To illustrate the results, consider the arithmetical semigroup G with primes $p_n \in P$, $n = 1, 2, \dots$, and with $\partial(p_n) = \ln(\frac{n+\rho}{\rho})$. Then

$$\pi_G^{\#}(x) = [\rho e^x - \rho] = \rho e^x + R(x), \quad -\rho \leq R(x) \leq -\rho + 1.$$

We compare the exact values of $\mathcal{N}_{G,k}^{\#}(x)$ and the asymptotic values given by Theorem 1 for $\rho = 0.5, 1$, and 2 at the points $x = 1, 2, \dots, 7$. The results are presented in Fig. 1 for $k = 2$ and $k = \infty$ (the Fermi and Bose cases). We also present the dependence of $\mathcal{N}_{G,k}^{\#}$ on the parameter $k \in [2, 8]$ at the point $x = 7$.

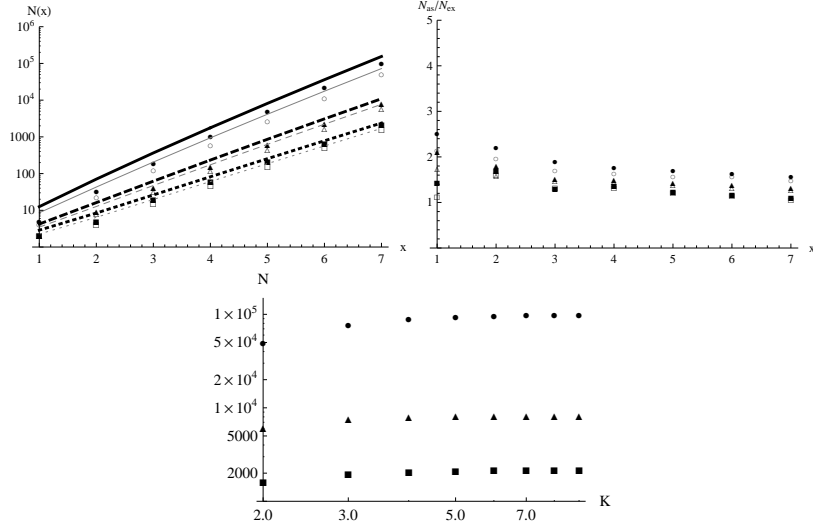


FIGURE 1. Left: the comparison of exact (points) and asymptotic (lines) values of $\mathcal{N}_{G,k}^\#(x)$ for $k = \infty$ (solid points and black lines) and $k = 2$ (empty points and gray lines). The parameter values are $\rho = 0.5, 1, 2$ (dotted lines and squares, dashed lines and triangles, and solid lines and circles, respectively).

Right: the ratio of the asymptotic values to the exact values of $\mathcal{N}_{G,k}^\#$ for $k = 2, \infty$ and $\rho = 0.5, 1, 2$ with the same notation.

Bottom: the dependence of $\mathcal{N}_{G,k}^\#(x)$ on the parameter $k \in [2, 8]$ at the point $x = 7$ for $\rho = 0.5, 1, 2$ (squares, triangles, and circles, respectively).

The figure illustrates the convergence of asymptotic formulas to the exact values (right). It also illustrates the fact, that the rate of growth $\mathcal{N}_{G,k}^\#(x)$ with x is the same for all k and asymptotics $\mathcal{N}_{G,k}^\#(x)$ differs by a factor (left). This fact follows from corollaries. We also can see from the bottom figure that this factor tends to 1 very rapidly with increasing parameter k and the value $k = 10$ can already be treated as “infinity” from the viewpoint of convergence.

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